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ON VORTEX PERTURBATIONS IN A FREE-INTERACTING BOUNDARY LAYER†

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Large-scale structures with an inviscid, non-linear subdomain (deck) on the bottom of a boundary layer in the case of subsonic and transonic free stream velocities are considered. A class of locally inviscid perturbations with an internal line of discontinuity of the tangential velocity, which leads to the appearance of a free term on the right-hand side of the Benjamin–Ono equations, is investigated. The shape of the above-mentioned line is sought and it is determined from the solution of a system of one-dimensional non-stationary equations in which, apart from the Benjamin–Ono equation, a kinematic condition and an equation for the inviscid deck close to the wall also occur. An example of a periodic, non-linear solution is constructed and amplitude constraints which ensure its realization are formulated.

1. ESTIMATES OF THE SCALES OF THE PERTURBATIONS. ASYMPTOTIC EXPANSIONS IN THE MAIN PART OF THE BOUNDARY LAYER AND IN THE POTENTIAL FLOW DOMAIN

Suppose the profiles of the longitudinal component of the velocity u^* and the gas density ρ^* are specified by the functions $U_0(Y_m)U_\infty^*$ and $R_0(Y_m)\rho_\infty^*$ in a boundary layer at a distance L^* from the leading edge of the flat plate. In these functions, the coordinate, normal to the plate, is $Y_m = \varepsilon^{-4}y^*L^{*-1}$, $\varepsilon = \mathrm{Re}^{-1/8}$ and the Reynolds number $\mathrm{Re} = \rho_\infty^* U_\infty^* L^*/\mu_\infty^* \to \infty$. Henceforth, dimensional quantities are denoted by asterisks and the subscript ∞ refers to parameters of the unperturbed flow at infinity. The origin of the Cartesian system of coordinates x^*, y^* is placed at the leading edge and we assume that the free stream of gas with velocity U_∞^*, Mach number M_∞ and viscosity coefficient μ_∞^* is directed along the x^* axis.

If the longitudinal component of the velocity $U_0U_\infty^*$ is perturbed by an amount of the order of αU_∞^*, $\alpha \ll 1$ in the domain of the boundary layer of the length $x^* - L^* = \ell L^*$, $\ell \ll 1$, then, according to the continuity equation, the perturbation of the vertical component of the velocity υ^* is of the order of $\alpha\varepsilon^4\ell^{-1}U_\infty^*$. This estimate holds when the inner variable of the boundary layer $Y_m = O(1)$. The excess pressure $p^* - p_\infty^*$, which is induced by the increase in the displacement thickness, in the potential flow outside the boundary layer ($Y_m \gg 1$) is estimated as $\alpha\varepsilon^4\ell^{-1}\rho_\infty^* U_\infty^{*2}$. On the other hand, in the non-linear part of the boundary layer close to the wall $Y_m \ll 1$ or, more precisely, $Y_m = O(\alpha)$, $U_0 = O(\alpha)$, the order of the excess pressure $\alpha^2\rho_\infty^* U_\infty^{*2}$ is determined by the quadratic terms in the equations of motion and the time t^* has a characteristic value $\ell\alpha^{-1}L^*U_\infty^{*-1}$. The relation $\ell = \varepsilon^4\alpha^{-1}$ is obtained from this. Note that, at short distances of the order of ℓL^*, the dependence of the functions U_0 and R_0 on the coordinate x^* is unimportant and will not subsequently be taken into consideration.

Hence, for the class of perturbed motions in the main deck of the boundary layer (deck 2) which are being considered, the expansions of the gas parameters in asymptotic sequences has the form

$$\frac{u^*}{U_\infty^*} = U_0 + \alpha u_{1m} + \alpha^2 u_{2m} + \ldots, \quad \frac{\upsilon^*}{U_\infty^*} = \alpha^2\upsilon_{1m} + \alpha^3\upsilon_{2m} + \ldots$$
$$\frac{\rho^*}{\rho_\infty^*} = R_0 + \alpha\rho_{1m} + \alpha^2\rho_{2m} + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \alpha^2 p_{1m} + \alpha^3 p_{2m} + \ldots \tag{1.1}$$

and the arguments T, X, Y_m of the required functions $u_{jm} = u_{jm}(T, X, Y_m)$, $\upsilon_{jm}(T, X, Y_m)$, $\rho_{jm} = \rho_{jm}(T, X, Y_m)$, $p_{jm} = p_{jm}(T, X, Y_m)$ ($j = 1, 2, \ldots$) are defined by the expressions

$$t^* = \frac{L^*}{U_\infty^*}\left(1 + \frac{\varepsilon^4}{\alpha^2}T\right), \quad x^* = L^*\left(1 + \frac{\varepsilon^4}{\alpha}X\right), \quad y^* = L^*\varepsilon^4 Y_m \tag{1.2}$$

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We substitute expressions (1.1) and (1.2) into the system of Navier–Stokes equations. Some of the first terms of the above-mentioned expansion will then satisfy equations which are identical in accuracy with those derived for the main deck of the triple-deck theory of free interaction [1–5]. When account is taken of the second-approximation terms which have been written out above, we obtain

$$u_{1m} = A_1(T,X)\frac{dU_0}{dY_m}, \quad v_{1m} = \frac{\partial A_1}{\partial X}U_0(Y_m)$$

$$\rho_{1m} = A_1(T,X)\frac{dR_0}{dY_m}, \quad p_{1m} = p_{1m}(T,X)$$

$$u_{2m} = -M_\infty^2 U_0 p_{1m} - \int_{-\infty}^{X} \frac{\partial v_{2m}}{\partial Y_m} dX' \tag{1.3}$$

$$v_{2m} = -\frac{\partial A_1}{\partial T} - A_1\frac{\partial A_1}{\partial X_m}\frac{dU_0}{dY_m} - Y_m U_0(M_\infty^2 - 1)\frac{\partial p_{1m}}{\partial X} -$$

$$-U_0\frac{\partial p_{1m}}{\partial X}\int_{Y_m}^{\infty}\left[\frac{1}{R_0(Y_m')U_0^2(Y_m')} - 1\right]dY_m' - \frac{\partial A_2}{\partial X}U_0(Y_m)$$

$$\rho_{2m} = M_\infty^2 R_0 p_{1m} + \frac{1}{2}A_1^2 U_0\frac{d}{dY_m}\left(U_0^{-1}\frac{dR_0}{dY_m}\right) - \int_{-\infty}^{X}\frac{\partial R_0}{\partial Y_m}\left(\frac{\partial A_1}{\partial T} + v_{2m}\right)dX'$$

where the functions $A_1 = A_1(T, X), A_2 = A_2(T, X)$ are arbitrary.

The asymptotic form of the solution of (1.1) in the upper edge of the boundary layer follows from (1.3) and, taking into account the limit properties $U_0 \to 1, R_0 \to 1$, we find, when $Y_m \to \infty$

$$\frac{u^*}{U_\infty^*} \to 1 - \alpha^2 p_{1m} + \ldots, \quad \frac{v^*}{U_\infty^*} \to 1 - \alpha^2\frac{\partial A_1}{\partial X} + \ldots$$
$$\frac{\rho^*}{\rho_\infty^*} \to 1 + \alpha^2 p_{1m} + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} \to \alpha^2 p_{1m} + \ldots \tag{1.4}$$

As far as the external potential flow is concerned, in the deck 1 which is adjacent to the boundary layer from above, the following expansions hold

$$\frac{u^*}{U_\infty^*} = 1 + \alpha^2 u_{1u} + \alpha^3 u_{2u} + \ldots, \quad \frac{v^*}{U_\infty^*} = \alpha^2 v_{1u} + \alpha^3 v_{2u} + \ldots$$

$$\frac{\rho^*}{\rho_\infty^*} = 1 + \alpha^2\rho_{1u} + \alpha^3\rho_{2u} + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \alpha^2 p_{1u} + \alpha^3 p_{2u} + \ldots \tag{1.5}$$

The arguments T and X of the required functions $u_{ju} = u_{ju}(T, X, Y_u)$, $v_{ju} = v_{ju}(T, X, Y_u)$, $\rho_{ju} = \rho_{ju}(T, X, Y_u)$, $p_{ju} = p_{ju}(T, X, Y_u)$ $(j = 1, 2, \ldots)$ are specified by relations (1.2), and the variable Y_u is introduced by the equality

$$y^* = L^*\varepsilon^4\alpha^{-1}Y_u \tag{1.6}$$

The form of expansions (1.5) is dictated by the asymptotic formulae (1.4) and, moreover, $Y_u = \alpha Y_m$. The representation of the solution of the Navier–Stokes equations by means of (1.5) and (1.6) gives

$$\frac{\partial u_{1u}}{\partial X} + \frac{\partial \rho_{1u}}{\partial X} + \frac{\partial v_{1u}}{\partial Y_u} = 0, \quad \frac{u_{1u}}{\partial X} + \frac{\partial p_{1u}}{\partial X} = 0$$
$$\frac{\partial v_{1u}}{\partial X} + \frac{\partial p_{1u}}{\partial Y_u} = 0, \quad \frac{\partial p_{1u}}{\partial X} = \frac{1}{M_\infty^2}\frac{\partial \rho_{1u}}{\partial X} \tag{1.7}$$

Hence

$$\left(1-M_\infty^2\right)\frac{\partial p_{1u}}{\partial X}=\frac{\partial v_{1u}}{\partial Y_u},\quad \frac{\partial p_{1u}}{\partial Y_u}=-\frac{\partial v_{1u}}{\partial X}$$

and the function $p_{1u}(T, X,(1-M_\infty^2)^{-1/2}Y_s)+i(1-M_\infty^2)^{-1/2}\upsilon_{1u}(T, X,(1-M_\infty^2)^{-1/2}Y_s)$ is an analytic function of the complex variable $X + iY_s$. The Schwartz integral for the half-plane, as it applies to the above-mentioned analytic function, has the form

$$p_{1u}(T,X,0)=\frac{1}{\pi}(1-M_\infty^2)^{-1/2}\int_{-\infty}^{\infty}\frac{v_{1u}(T,\xi,0)}{\xi-X}d\xi \tag{1.8}$$

Here, the integral is understood in the sense of the principal Cauchy value.

If $Y_u \to 0$, then $Y_m \to \infty$ and, by virtue of (1.4), matching of the asymptotic expansions (1.1) and (1.5) lead to the conditions

$$p_{1u}(T,X,0)=p_{1m}(T,X),\qquad v_{1u}(T,X,0)=-\frac{\partial A_1}{\partial X} \tag{1.9}$$

The function to be determined $A_1 = A_1(T, X)$, together with the pressure p_{1m} (T, X), characterizes the magnitude of the instantaneous displacement of the streamlines in the boundary layer with respect to their unperturbed position when $X \to -\infty$. The perturbation of the pressure in the outer, potential flow domain 1 is induced by an increase in the displacement thickness, which is described by the function A_1 and is transmitted, according to the first of formulae (1.9), into the main deck of the boundary layer and, in turn, affects its displacing action. This mechanism finds reflection in the interaction boundary condition [1–5], which formally follows from (1.8) and (1.9)

$$p_{1m}=-\frac{1}{\pi}(1-M_\infty^2)^{-1/2}\int_{-\infty}^{\infty}\frac{\partial A_1(T,\xi)/\partial\xi}{\xi-X}d\xi \tag{1.10}$$

2. ASYMPTOTIC EXPANSIONS IN THE NON-LINEAR REGION ON THE BOTTOM OF THE BOUNDARY LAYER

When $Y_m = O(\alpha)$, the velocity perturbation becomes of the order of the velocity itself. Actually, in the unperturbed boundary layer

$$U_0=\lambda_1 Y_m+\ldots,\qquad R_0=r_0+\ldots,\qquad Y_m\to 0 \tag{2.1}$$

The expression for υ_{2m} from (1.3) contains an integral which diverges at the point $Y_m = 0$. The behaviour of this integral in the neighbourhood of the given point is determined not only by λ_1, r_0 but also depends on the higher terms of the expansion of the functions U_0, R_0 using Taylor's formula. We will therefore refine formulae (2.1), assuming that the surface is thermally isolated and, consequently, the equalities $d^2U_0(0)/dY_m^2 = 0$, $dR_0(0)/dY_m = 0$ hold. Then, by virtue of (1.3), expansion (1.1) in the limit as $Y_m \to 0$ can be written as follows:

$$\begin{aligned}
&\frac{u^*}{U_\infty^*}=\lambda_1 Y_m+\alpha\lambda_1 A_1+O(\alpha^2)\\
&\frac{v^*}{U_\infty^*}=-\alpha^2\lambda_1\frac{\partial A_1}{\partial X}Y_m-\alpha^3\left[\frac{\partial A_1}{\partial T}+\lambda_1 A_1\frac{\partial A_1}{\partial X}+\frac{1}{r_0\lambda_1}\frac{\partial p_{1m}}{\partial X}\right]+O(\alpha^4)\\
&\frac{\rho^*}{\rho_\infty^*}=r_0+O(\alpha^2)
\end{aligned} \tag{2.2}$$

Comparison of the first two terms in the expression for u^* from (2.2) leads to the estimate given above for the magnitude of the transverse coordinate Y_m, which establishes the lower boundary of deck 2. We now introduce deck 3 with a thickness of $\alpha\varepsilon^4 L^*$, where non-linear effects predominate, which is contiguous to deck 2 from below. The gas parameters in this domain can be represented in the form

$$\frac{u^*}{U_\infty^*} = \alpha u_{1a} + \alpha^2 u_{2a} + \ldots, \quad \frac{v^*}{U_\infty^*} = \alpha^3 v_{1a} + \alpha^4 v_{2a} + \ldots \tag{2.3}$$
$$\frac{\rho^*}{\rho_\infty^*} = \rho_{1a} + \alpha\rho_{2a} + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \alpha^2 p_{1a} + \alpha^3 p_{2a} + \ldots$$

and the unknown functions $u_{ja} = u_{ja}(T, X, Y_a)$, $v_{ja} = v_{ja}(T, X, Y_a)$, $\rho_{ja}(T, X, Y_a)$, $p_{ja} = p_{ja}(T, X, Y_a)$ $(j = 1, 2, \ldots)$ contain the variable $Y_a = O(1)$ as one of the arguments. The expression defining the variable Y_a

$$y^* = L^*\alpha\varepsilon^4 Y_a \tag{2.4}$$

gives the obvious relation $Y_m = \alpha Y_a$.

The system of equations

$$\frac{\partial\rho_{1a}}{\partial T} + \frac{\partial\rho_{1a}u_{1a}}{\partial X} + \frac{\partial\rho_{1a}v_{1a}}{\partial Y_a} = 0, \quad \frac{\partial p_{1a}}{\partial Y_a} = 0$$
$$\rho_{1a}\frac{\partial u_{1a}}{\partial T} + \rho_{1a}u_{1a}\frac{\partial u_{1a}}{\partial X} + \rho_{1a}v_{1a}\frac{\partial u_{1a}}{\partial Y_a} = -\frac{\partial p_{1a}}{\partial X} + \frac{\varepsilon^4}{\alpha^4}\frac{\partial}{\partial Y_a}\left(\mu\frac{\partial u_{1a}}{\partial Y_a}\right) \tag{2.5}$$
$$\frac{\partial\rho_{1a}}{\partial T} + u_{1a}\frac{\partial\rho_{1a}}{\partial X} + v_{1a}\frac{\partial\rho_{1a}}{\partial Y_a} = -\frac{\varepsilon^4}{\alpha^4}\rho_{1a}\frac{\partial}{\partial Y_a}\left(\frac{\mu}{\Pr}\frac{\partial}{\partial Y_a}\frac{1}{\rho_{1a}}\right)$$

follows from representation (2.3).

Here, $\mu = \mu^*\mu_\infty^{*-1}$ is the first coefficient of viscosity and Pr is the Prandtl number.

The limiting conditions when $Y_a \to \infty$, which the solutions of system of equations (2.5) must satisfy, follow from asymptotic expressions, one changes from the variable Y_m to the variable Y_a. In particular, the property

$$\frac{\rho^*}{\rho_\infty^*} = \rho_{1a} + O(\alpha^2)$$

holds when $X \to -\infty$ and $Y_a \to \infty$ (under the assumptions made in deriving (2.2)) and this enables us to assume [2, 6] that

$$\rho_{1a} = r_0, \quad \rho_{2a} = 0$$

everywhere in the non-linear deck 3.

The triple-deck model of free interaction [1–5], constructed to describe different types of motions of a liquid and gas under conditions when problems regarding the viscous flow domain and an outer inviscid flow do not split and must be solved simultaneously, is based on asymptotic expansion with respect to a small parameter $\varepsilon = \mathrm{Re}^{-1/8}$. The elements of the approach considered here which are new compared with the theory proposed earlier are the introduction of a parameter α, which is independent of the Reynolds number and can be of a smaller order than the parameter ε, and a different normalization of the variables t^*, x^*, y^*. If one puts $\alpha = \varepsilon$ in expansions (1.1), (1.5) and (2.3) for decks 1, 2 and 3 then, as can be seen from Eqs (2.5), the viscous tangential stresses in domain *3* become so important that they are also the mechanism of the non-linearity. In this case, we return to the triple-deck theory [1–5] which enables us to reduce the problem of an interacting boundary layer to the solution of system of equations (2.5) with the boundary conditions

$$Y_a = 0: \quad u_{1a} = v_{1a} = 0; \quad Y_a \to \infty: \quad u_{1a} \to \lambda_1(Y_a + A_1)$$

and the additional condition (1.10) with $p_{1m} = p_{1a}$ and $Y_m = Y_a$.

3. SPLITTING OF THE NON-LINEAR DOMAIN ON THE BOTTOM INTO TWO SUBLAYERS IN THE CASE OF COMPARATIVELY LARGE AMPLITUDE PERTURBATIONS

We will now consider the asymptotic structure of a perturbed flow, assuming that the inequalities

$$\varepsilon \ll \alpha \ll 1 \tag{3.1}$$

are satisfied.

In this case, as can be seen from Eqs (2.5), the non-linear part is split into a main inviscid sublayer $Y_a = O(1)$ and a viscous sublayer, adjacent to the wall, of thickness $Y_a = O(\alpha^{-2}\varepsilon^2)$, and expression (2.4) still holds for the variable Y_a. As previously, we mean by deck *3* of the quadruple-deck scheme of the perturbed velocity field which arises, the sublayer $Y_a = O(1)$ in which expansions (2.3) hold and in which terms in Eqs (2.5) containing the viscosity coefficient μ can be neglected. The lower boundary $Y_a \to 0$ of deck 3 serves as the outer edge of deck 4 which comes into contact with the plate surface and for which the characteristic vertical coordinate $Y_\ell = O(1), y^* = L^*\alpha^{-1}\varepsilon^6 Y_\ell$ is introduced. The expansions of the stream functions in deck 4

$$\frac{u^*}{U_\infty^*} = \alpha u_{1\ell}(T, X, Y_\ell) + \ldots, \quad \frac{v^*}{U_\infty^*} = \alpha\varepsilon^2 v_{1\ell}(T, X, Y_\ell) + \ldots$$

$$\frac{\rho^*}{\rho_\infty^*} = r_0 + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \alpha^2 p_{1\ell}(T, X, Y_\ell) + \ldots \tag{3.2}$$

leads to a system of Prandtl equations with the boundary conditions

$$Y_\ell = 0 \quad : \quad u_{1\ell} = v_{1\ell} = 0 \tag{3.3}$$

$$Y_\ell \to \infty \quad : \quad u_{1\ell} \to u_{1a}(T, X, 0) \tag{3.4}$$

Boundary-value problems for the principal terms of the expansions of the gas parameters in decks 3 and 4 will not contain the constants $\lambda_1, r_0, s = (M_\infty^2 - 1)^{1/2}$ nor the viscosity coefficient $\mu_0 = \mu(r_0^{-1})$ calculated at the temperature $R_0^{-1}(Y_m)$ of the unperturbed boundary layer when $Y_m = 0$, if the following similitude transformation is carried out [1–5]

$$\begin{gathered} t = b^t T, \quad x = b^x X, \quad y \equiv y_a = b^y Y_a, \quad y_\ell = b^y = Y_\ell, \quad A = b^y A_1 \\ u = b^u u_{1a}, \quad v = b^v v_{1a}, \quad u_\ell = b^u u_{1\ell}, \quad v_\ell = b^v v_{1\ell}, \quad p = b^p p_{1m} \end{gathered} \tag{3.5}$$

where

$$\begin{gathered} b^t = \lambda_1^{3/2}\mu_0^{1/2}s^{1/2}, \quad b^x = \lambda_1^{5/4} r_0^{1/2}\mu_0^{1/4}s^{3/4} \\ b^y = \lambda_1^{3/4} r_0^{1/2}\mu_0^{-1/4}s^{1/4}, \quad b^p = \lambda_1^{-1/2}\mu_0^{-1/2}s^{1/2} \\ b^u = \lambda_1^{-1/4} r_0^{1/2}\mu_0^{-1/4}s^{1/4}, \quad b^v = \lambda_1^{-3/4} r_0^{1/2}\mu_0^{-3/4}s^{-1/4} \end{gathered} \tag{3.6}$$

In the variables (3.5) and (3.6), the system of equations (2.5) for the non-linear inviscid deck 3, subject to assumption (3.1), can be rewritten as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial p}{\partial y} = 0 \tag{3.7}$$

On replacing the variable Y_m in (2.2) by the variable $Y_a = \alpha^{-1}Y_m$, we obtain the limit form of the expansions (2.3) when $Y_a \to \infty$. In particular, the leading terms (with respect to the parameter α) of expressions (2.2) specify the asymptotic form of the functions $u_{1a}, v_{1a}, \rho_{1a}, p_{1a}$ on the upper edge of deck *3*. As can be seen from (2.2), the above-mentioned functions, rewritten in the special system of units (3.5), (3.6), possess the following behaviour as $y \to \infty$

$$u = y + A, \quad v = -\frac{\partial A}{\partial x}y - \frac{\partial A}{\partial t} - A\frac{\partial A}{\partial x} - \frac{\partial p}{\partial x} \tag{3.8}$$

Expressions (3.8), which establish the asymptotic boundary conditions for system of equations (3.7) satisfy this system identically. Hence, functions (3.8) can be continued into the domain of finite values of y as solutions of system (3.7) and the conditions for the matching of the asymptotic expansions in decks 2 and 3 in the limit as $y \to \infty$ will be satisfied.

We now return to the matching of the asymptotic expansions in decks 3 and 4 (assuming that there are no irregularities with a vertical dimension of the order of $\alpha\varepsilon^4$ on the surface of the plate around which the flow occurs). The relation $Y_\ell = \alpha^2\varepsilon^{-2}Y_a$ follows from the definition of the transverse coordinate Y_ℓ for the lowest deck 4 which is adjacent to the plate and, by virtue of (3.1), the variable $Y_\ell \to \infty$ when $Y_a \to 0$. The internal limit of expansions (2.2) as $Y_a \to 0$ must be identical with the external limit of expansions (3.2) and $Y_\ell \to \infty$. This means, in particular, that the function υ_{1a} (and, changing to the variables (3.5) and (3.6), also the function υ) must be of the order of $\alpha^{-2}\varepsilon^2$ when $y = O(\alpha^{-2}\varepsilon^2)$. If the function υ is taken from solution (3.8) of system of equations (3.7), then the above-mentioned limit property can only be satisfied when the condition

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial^2 A(t,\xi)/\partial \xi^2}{\xi - x}d\xi \tag{3.9}$$

is satisfied, since the functions p and A are independent of the variable y and relation (1.10) between them, in the system of units (3.5), (3.6), acquires the form

$$p = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial A(t,\xi)/\partial \xi}{\xi - x}d\xi \tag{3.10}$$

One of the conditions for the matching of the leading terms of expansions (2.3) and (3.2), which is expressed by Eq. (3.9) (known as the Benjamin–Ono equation [7, 8]) is equivalent to impermeability condition $\upsilon = 0$ when $y = 0$ by virtue of (3.8). In the triple-deck theory [1–5], if the displacement thickness A and the self-induced pressure p can only be determined when solving the boundary-value problem in the viscous sublayer adjacent to the wall, the quadruple-deck scheme reduces the construction of the two-dimensional flow pattern in decks 1–3 to the solution of one-dimensional equation (3.9). The flow fields in these decks are established using the function A by means of formulae (1.3), (3.8) and (3.10). As far as the viscous deck 4 close to the wall is concerned, the function $u_{1a}(T, X, 0)$, which occurs in asymptotic condition (3.4), after the change of variables (3.5) and (3.6), is identical with $A = A(t, x)$ as can be seen from (3.8). In this sense, deck 4, which is close to the wall, plays a passive role since the flow parameters in it are determined using the pressure and velocity distributions on its outer edge, which are found independently.

The possibility, which has been pointed out earlier [9], of describing the mechanism of the interaction on the basis of the Benjamin–Ono equation (or Burger's equation in the case of a supersonic flow velocity in the deck 1) and the realization of a quadruple-deck structure is a consequence of inequality (3.1); passing to the limit of high frequency pulsations within the framework of a triple-deck scheme [1–5], which is accompanied by stratification of the lower deck close to the wall, also leads to the above-mentioned equations [10, 11]. The modification of the theory under consideration in order to apply it to the problem of the flow past a small irregularity on a plate with a vertical dimension of the order of $\alpha\varepsilon^4$ reduces to the appearance of a free term on the right-hand side of Eq. (3.9). It has previously been shown [11, 12] that the free term serves as a source of self-excited oscillations in the form of soliton solutions even in the case of steady-state boundary conditions.

The steady-state analogue of system of equations (3.7) describes one of the subdomains in the asymptotic problem of the reattachment of a supersonic flow [13] and, also, the neighbourhood of the separation point of the boundary layer on a wall which is moving downstream [14]. Supersonic flow past a plate with periodically oscillating screen and the propagation of the perturbations, caused by a sudden change in the bottom pressure, along the surface of a wedge have been treated in [15] as examples of unsteady locally inviscid flows with interaction.

4. THE STRUCTURE OF DECK 3 WHEN THERE IS A TANGENTIAL DISCONTINUITY IN THE VELOCITY

We will now consider a moving contour defined by the equation

$$y^* = \alpha\varepsilon^4 L^* G_1(T, X) \tag{4.1}$$

T and X are defined by formula (1.2) and the components of its outward normal are

$$n_x = -\frac{\varkappa}{1+\varkappa^2}, \quad n_y = \frac{1}{1+\varkappa^2}; \quad \varkappa = \frac{\partial y^*}{\partial x^*} = \alpha^2 \frac{\partial G_1}{\partial X}$$

The velocity of the contour along the normal to it, α^5, can be replaced, apart from quantities of the order of υ_n^*, by a derivative of function (4.1) with respect to the time t^* (which is of the order of α^3).

We set up an impermeability condition on the contour (4.1), assuming the contour is a streamline. Since it is a distance from the solid wall which is equal in order of magnitude to the thickness of the inviscid non-linear deck *3* in which the coordinate $Y_a = O(1)$ is specified by means of (2.4), then, subject to the impermeability condition $u^* n_x + \upsilon^* n_y = \upsilon_n^*$, functions of the first approximation of the asymptotic series (2.3), confining ourselves to the leading terms of the expansions of the components of the normal n_x, n_y with respect to the parameter α, have to be taken as the gas velocities u^*, υ^*. Hence, the kinematic condition on the line (4.1) in the special system of units (3.5), (3.6) for deck 3 takes the form

$$y = G \quad : \quad \frac{\partial G}{\partial t} = v + u\frac{\partial G}{\partial x} \tag{4.2}$$

where $G(t, x) = b^y G_1(T, X)$ and the coefficient b^y is defined in (3.6).

We shall assume that solution (3.8) obtained by analytic continuation from deck 2 holds in the subdomain $G(t,x) < y < +\infty$ of deck 3 up to its lower boundary $y = G(t, x)$. Substitution of expressions (3.8) into (4.2) then leads to the relation

$$\frac{\partial(A+G)}{\partial t} + (A+G)\frac{\partial(A+G)}{\partial x} + \frac{\partial p}{\partial x} = 0 \tag{4.3}$$

We will denote by $\mathcal{H}$ an operator, the action of which on the function $f(t, x)$ is expressed in terms of a Cauchy-type integral

$$\mathcal{H}\{f\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f d\xi}{\xi - x}$$

By virtue of (3.10), condition (4.3) has, as a consequence, the Benjamin–Ono equation

$$\frac{\partial \tilde{A}}{\partial t} + \tilde{A}\frac{\partial \tilde{A}}{\partial x} = \mathcal{H}\left\{\frac{\partial^2 \tilde{A}}{\partial x^2}\right\} - \phi(t, x) \tag{4.4}$$

in the function $\tilde{A} = A + G$ and the free term on the right-hand side of (4.4) is determined by the function $G = G(t, x)$ as follows:

$$\phi(t, x) = \mathcal{H}\left\{\frac{\partial^2 G}{\partial x^2}\right\} \tag{4.5}$$

Suppose the surface of the body around which the flow occurs (which is a flat plate) has an irregularity with a vertical dimension of the order of $\alpha\varepsilon^4 L^*$. Then, in the variables in which equations (3.7) are written, the shape of the solid surface can be defined by the equation $y = F(x)$. Equations (3.7) imply that the vorticity

$$\omega = \frac{\partial u}{\partial y}$$

is conserved along the streamline

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0$$

We will now consider the solution of the system of equations (3.7) with a constant vorticity $\omega = \Omega_0$ in the subdomain $F(t,x) < y < G(t,x)$ of deck 3

$$u = \Omega_0 y + \Theta(t,x) \tag{4.6}$$

The impermeability condition on the solid surface $y = F(x)$ leads to the expression

$$v = -y\frac{\partial\Theta}{\partial x} + \frac{\partial F\Theta}{\partial x} + \frac{\Omega_0}{2}\frac{\partial F^2}{\partial x} \tag{4.7}$$

We substitute (4.6) and (4.7) into the kinematic condition (4.2); then

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left[(G-F)\Theta + \frac{\Omega_0}{2}(G^2 - F^2)\right] = 0 \tag{4.8}$$

The pressure gradient is established using the first equation of system (3.7)

$$\frac{\partial\Theta}{\partial t} + \Theta\frac{\partial\Theta}{\partial x} + \Omega_0\frac{\partial}{\partial x}\left(F\Theta + \frac{\Omega_0}{2}F^2\right) = -\frac{\partial p}{\partial x} \tag{4.9}$$

On the line $y = G(t,x)$, the velocity component which is tangential to it, generally speaking, suffers a discontinuity.

5. SOME EXACT SOLUTIONS OF THE BENJAMIN–ONO EQUATION

Suppose $A(t,x)$ is the solution of the homogeneous Benjamin–Ono equation (3.9). It can be verified that, for any value of the parameter ρ, the function

$$A_\rho(t,x) = (1-\rho)A[(1-\rho)t, x]$$

satisfies the Benjamin–Ono equation with the inhomogeneous right-hand side

$$\frac{\partial A_\rho}{\partial t} + A_\rho\frac{\partial A_\rho}{\partial x} = \mathcal{H}\left\{\frac{\partial^2 A_\rho}{\partial x^2}\right\} - \phi(t,x)$$

if the free term

$$\phi(t,x) = \mathcal{H}\left\{\frac{\partial^2 B_\rho}{\partial x^2}\right\}, \quad B_\rho(t,x) = \rho A_\rho(t,x)$$

We now consider the solution, which is periodic with a spatial wavelength of $2\pi k^{-1}$ and a frequency kc, of the homogeneous Benjamin–Ono equation

$$A(t,x) = \frac{2k^2}{c}\left\{1 - \left[1 - \frac{k^2}{c^2}\right]^{1/2}\cos[k(x-ct)]\right\}^{-1} \tag{5.1}$$

which depends on two parameters: the wave number k and the phase velocity c. The function (5.1) is even with respect to k, and we shall therefore assume that $k > 0$. As far as the sign of c is concerned, expression (5.1) only specifies the solution of the Benjamin–Ono equation when $c < 0$ and therefore represents a wave which is always travelling upstream.

If $A(t,x)$, is the solution of the Benjamin–Ono equation, then

$$\mathcal{G}A(\mathcal{G}^2 t, \mathcal{G}x) \tag{5.2}$$

and

$$\mathcal{D} + A(t, x - \mathcal{D}t) \tag{5.3}$$

are also the solutions at any $\mathcal{G}$ and $\mathcal{D}$.

Application of transformation (5.2) to the periodic solution (5.1) of the Benjamin–Ono equation is equivalent to the changes in the wave number and phase velocity: $k \to \mathcal{G}k$, $c \to \mathcal{G}c$, and, hence, does not increase the number of parameters occurring in this solution.

Transformation (5.3) (the change to a system of coordinates moving at a velocity $-\mathcal{D}$) leads to the three-parameter solution

$$A(t,x) = \mathcal{D} - \frac{2k^2}{|c|}\left\{1 - \left[1 - \frac{k^2}{|c|^2}\right]^{1/2} \cos[k(x + (|c| - \mathcal{D})t)]\right\}^{-1} \tag{5.4}$$

We recall that, in (5.1), $c < 0$ always. In view of (5.2), (5.3) can be considered, without loss of generality, as the steady-state solution with a period of 2π, on putting $\mathcal{D} = |c|$, $k = 1$ and confining ourselves to a single parameter which is taken as

$$0 < \varepsilon_0 = \left(1 - \frac{k^2}{|c|^2}\right)^{1/2} < 1$$

Hence the function

$$A(t, x) = A_0(x) = \frac{1}{(1 - \varepsilon_0^2)^{1/2}}\left[1 - \frac{2(1 - \varepsilon_0^2)}{1 - \varepsilon_0 \cos x}\right] \tag{5.5}$$

is the solution of the homogeneous Benjamin–Ono equation. The extremum values of the function (5.5) at the points $x = 0$ and $x = \pi$ are

$$A_{\min} = -\frac{1 + 2\varepsilon_0}{(1 - \varepsilon_0)^{1/2}}, \quad A_{\max} = -\frac{1 - 2\varepsilon_0}{(1 - \varepsilon_0)^{1/2}} \tag{5.6}$$

Substituting the parameter δ_0, which is equal to the amplitude of the non-linear oscillations

$$\delta_0 = A_{\max} - A_{\min}, \quad \varepsilon_0 = \frac{\delta_0}{(\delta_0^2 + 16)^{1/2}}$$

into (5.5) instead of ε_0, we obtain a further representation of the 2π-periodic steady-state solution of the Benjamin–Ono equation

$$A_0(x) = \frac{(\delta_0^2 + 16)^{1/2}}{4} - \frac{8}{(\delta_0^2 + 16)^{1/2} - \delta_0 \cos x} \tag{5.7}$$

As $\delta_0 \to 0$, (5.7) defines a neutral Tollmien–Schlichting wave in a system of coordinates which moves downstream at a velocity $|c| \to 1 + 0$

$$A_0(x) = -1 - \frac{\delta_0}{2}\cos x + O(\delta_0^2)$$

When $\delta_0 \to \infty$, the limit form of (5.6)

$$A_{\max} = \frac{\delta_0}{4} + O\left(\frac{1}{\delta_0}\right), \quad A_{\min} = -\frac{3\delta_0}{4} + O\left(\frac{1}{\delta_0}\right)$$

shows that solution (5.7) is sign variable and the deviation of the function $A_0(x)$ downwards from a zero value is three times greater than the deviation upwards.

We will rewrite solution (5.7) under the assumption that the amplitude of the oscillations tends to infinity $\delta_0 \to \infty$

$$A_0(x) = \frac{\delta_0}{4} + \frac{2}{\delta_0} + O\left(\frac{1}{\delta_0^3}\right) - \frac{8}{\delta_0}\left[1 - \cos x + \frac{8}{\delta_0^2} + O\left(\frac{1}{\delta_0^4}\right)\right]^{-1} \tag{5.8}$$

In the case of $1 - \cos x = O(1)$, the asymptotic form of solution (5.7) for $\delta_0 \to \infty$ is defined by the expression

$$A_0(x) = \frac{\delta_0}{4} + O\left(\frac{1}{\delta_0}\right) \tag{5.9}$$

Hence, in the limit of large amplitudes which is being studied, solution (5.7) tends from below to a constant value equal to its maximum value. However, as can be seen from (5.8), the asymptotic representation (5.9) ceases to be valid in the narrow domains $1 - \cos x = O(\delta_0^{-2})$. As a consequence of the periodicity of the solution, we will confine ourselves to considering it in the neighbourhood of zero. On expanding the function $\cos x$ in (5.8) in a MacLaurin series up to $O(x^4)$ as $x \to 0$, we find

$$A_0(x) = \frac{\delta_0}{4} - \delta_0\left[1 + \frac{\delta_0^2}{16}x^2 + O\left(\frac{1}{\delta_0}\right)\right]^{-1} \tag{5.10}$$

The asymptotic form (5.10) of solution (5.7) is a solitary standing wave on a non-zero background $\delta_0/4$ (a soliton on a pedestal). Formula (5.10) holds in the domains $x \pm 2\pi n = O(\delta_0^{-1})$ ($n = 0, 1, 2, \ldots$), and therefore specifies the solution in the form of narrow tongues.

6. SPECTRAL DECOMPOSITION OF THE NON-LINEAR PERIODIC SOLUTION OF THE BENJAMIN-ONO EQUATION. EXPANSION IN A SYSTEM OF EQUIDISTANT SOLITONS

We shall now indicate other forms of representation of periodic solution (5.1). Suppose that

$$\operatorname{sh}\mathfrak{Q} = \frac{k}{|c|}\left(1 - \frac{k^2}{c^2}\right)^{-1/2}, \quad \operatorname{ch}\mathfrak{Q} = \left(1 - \frac{k^2}{c^2}\right)^{-1/2} \tag{6.1}$$

Then

$$\begin{aligned} A(t, x) &= -\frac{2k\operatorname{sh}\mathfrak{Q}}{\operatorname{ch}\mathfrak{Q} - \cos[k(x+|c|t)]} = -2\operatorname{Im}\left\{ik\frac{1+\exp[-\mathfrak{Q} - ik(x+|c|t)]}{1-\exp[-\mathfrak{Q} - ik(x+|c|t)]}\right\} = \\ &= -2\operatorname{Im}\left\{ik\left[1 + 2\sum_{N=1}^{\infty}\exp[-(\mathfrak{Q} + ik(x+|c|t))N]\right]\right\} = \\ &= -2k\left\{1 + 2\sum_{N=1}^{\infty}\left(\frac{|c|-k}{|c|+k}\right)^{N/2}\cos[kN(x+|c|t)]\right\} \end{aligned} \tag{6.2}$$

(when obtaining the last equality in the chain, the quantity $\exp(-\mathfrak{Q})$ was eliminated using (6.1) and the imaginary part was separated out).

It can be seen from the spectral composition (6.2) of the coherent soliton structure that the amplitudes of the spectral modes of the periodic solution (5.1) form a geometric progression in Fourier space. At the same time, in physical space, solution (5.1) decomposes into strongly non-linear localized structures for any relations between the parameters $|c|$ and k and not only in the case of formula (5.10) being described, for the limit $|c|/k \to \infty$ ($\varepsilon_0 \to 0$, $\delta_0 \to \infty$). This last assertion was formulated earlier [16] and is based on a consideration of the integral

$$I = \oint\left[\left(\eta - 2\frac{\pi}{k}\mathfrak{X}\right)^2 + \zeta^2\right]^{-1}\frac{d\mathfrak{X}}{\exp(2\pi i\mathfrak{X}) - 1} \tag{6.3}$$

along a closed contour Γ in the complex plane $\mathfrak{X}$.

Suppose the contour Γ is the circle $|\mathcal{X}| = R_N = N + 1/2$ ($N = 1, 2, \ldots$).

Then, on representing integral (6.3) in the form of a sum of the residues at the poles of the integrand and taking account of the fact that this sum tends to zero when $N \to \infty$, we obtain

$$-\frac{2k \operatorname{th} k\zeta}{1-\sqrt{1-\operatorname{th} k\zeta}\cos k\eta} = -\sum_{n=-\infty}^{\infty} 4\zeta\left[\left(\eta - 2\frac{n\pi}{k}\right)^2 + \zeta^2\right]^{-1} \tag{6.4}$$

The left-hand side of this equality when

$$\operatorname{th} k\zeta = \frac{k}{|c|}, \qquad \eta = x + |c|t$$

is the solution (5.1) of the Benjamin–Ono equation. The non-linear periodic solution (5.1) can be represented as a sum of solitons equidistant from one another

$$A(t,x) = -\frac{2k^2}{|c|}\left\{1 - \left[1 - \frac{k^2}{|c|^2}\right]^{1/2} \cos[k(x + |c|t)]\right\}^{-1} =$$

$$= -\sum_{n=-\infty}^{\infty} \frac{4|c|\theta}{1 + |c|^2\theta^2(x + |c|t - 2n\pi/k)^2}, \quad \theta = \frac{k}{|c|}\operatorname{arcth}^{-1}\frac{k}{|c|} \tag{6.5}$$

Note that, $\theta < 1$ always and it is therefore only in the limit $\theta \to 1$, which is attained when $k/|c| \to 0$, that each term of the sum (6.5) is an exact solution [7, 8]

$$A(t,x) = -\frac{4|c|}{1 + |c|^2(x + |c|t)^2} \tag{6.6}$$

of the Benjamin–Ono equation in the form of a solitary wave (a soliton). Solution (6.6) is obtained by applying the transformation $x \to x + |c|t$, $A \to A - |c|$ to solution (5.10) and, in the last solution, one has to put $\delta_0 = 4|c|$.

7. AN EXAMPLE OF A PERIODIC VORTEX STRUCTURE IN THE CASE OF A TANGENTIAL DISCONTINUITY IN THE FLOW

According to the remark made at the beginning of Section 5, the introduction of a line of tangential discontinuity in the velocity

$$y = G(t,x) = \rho\tilde{A}(t,x), \qquad \tilde{A}(t,x) = (1-\rho)A[(1-\rho)t, x] \tag{7.1}$$

enables one to assert that the equation

$$\frac{\partial\tilde{A}}{\partial t} + \tilde{A}\frac{\partial\tilde{A}}{\partial x} = -\frac{\partial p}{\partial x} \tag{7.2}$$

will be satisfied if

$$\frac{\partial p}{\partial x} = -\mathcal{H}\left\{\frac{\partial^2\tilde{A}}{\partial x^2}\right\} - \phi(t,x), \qquad \phi(t,x) = \mathcal{H}\left\{\frac{\partial^2 G}{\partial x^2}\right\} \tag{7.3}$$

where, in (7.1), $A(t,x)$ is any solution of the Benjamin–Ono equation (3.9).

We shall seek a solution of the system of equations (4.3), (4.8), (4.9) and (7.3) in the form of a travelling wave with a phase velocity c. Then

$$p = c\tilde{A} - \tilde{A}^2/2 + \mathcal{H}_1 \tag{7.4}$$

Henceforth, we will confine ourselves to the case when $F = F_0 =$ const. From (4.9), we have

$$p = c\Theta - \Theta^2/2 + \Omega_0 F_0 \Theta + \mathcal{K}_2 \tag{7.5}$$

and if we put

$$\Theta = \tilde{A} - \Omega_0 F_0 \tag{7.6}$$

then continuity of the pressure on the line of tangential discontinuity $y = G$ is ensured by the following choice of the constants $\mathcal{K}_1$, $\mathcal{K}_2$ from (7.4) and (7.5)

$$\mathcal{K}_1 - \mathcal{K}_2 = c\Omega_0 F_0 - \Omega_0^2 F_0^2/2$$

We now return to Eq. (4.8) in the function G. The assumption concerning the self-similarity of its solution being of the type of a travelling wave with velocity c establishes the relation

$$cG - \Theta G + \Theta F_0 - \Omega_0 G^2/2 = \mathcal{K}_3 \tag{7.7}$$

We assume that the constant $\mathcal{K}_3 = \Omega_0 F_0^2/2 - cF_0$, and relation (7.7) is then satisfied in the case of the function

$$G = \rho(\Theta - c) - F_0 \tag{7.8}$$

subject to the condition

$$\rho = -2/\Omega_0 \tag{7.9}$$

We now take function (5.1) as the solution of the Benjamin–Ono equation. Then, the periodic solution of Eq. (4.4)

$$\tilde{A}(t,x) = \frac{2k^2(1-\rho)}{c}\left\{1 - \left[1 - \frac{k^2}{c^2}\right]^{1/2} \cos[k(x - c(1-\rho)t)]\right\}^{-1} \tag{7.10}$$

corresponds to the free term (4.5) which is induced by a line of tangential discontinuity of the following form

$$y = G(t,x) = F_0 - c\rho(1-\rho)\left\{1 - \frac{2k^2}{c^2}\left\{1 - \left[1 - \frac{k^2}{c^2}\right]^{1/2} \cos[k(x - c(1-\rho)t)]\right\}^{-1}\right\} \tag{7.11}$$

The expression from (7.11) in the outer braces is negative in the case when

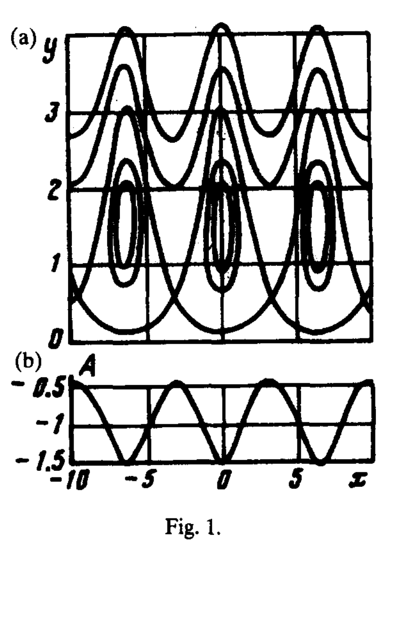

Fig. 1.

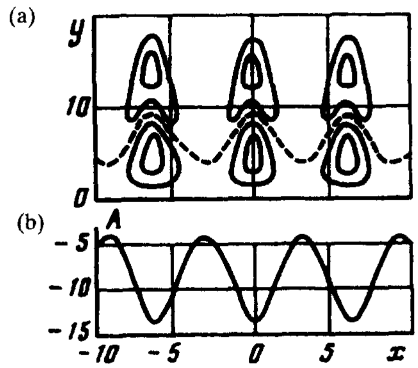

Fig. 2.

$$\sqrt{3}\,|c|/2 < k < |c| \tag{7.12}$$

For (7.12), the line of discontinuity does not intersect the solid wall if $\rho(1-\rho) < 0$, and, hence, the condition

$$\rho < 0 \text{ or } \rho > 1 \tag{7.13}$$

together with condition (7.12), ensures the realizability of the flow model with a line of tangential discontinuity of the velocity located in the stream.

A theoretical model of a recirculation zone with a characteristic length L^* of the order of magnitude of the chord of a thin body was proposed earlier [17]. Assuming that the width of the separation zone h^* does not exceed the body thickness, a system of equations was derived where the shape of the boundary (which is a contact discontinuity) of the vortex region was one of the unknown functions. It was shown that substantially non-linear unsteady perturbations lead to the breakdown, at a final instant of time, of this two-layer structure as a result of focussing and the ejection of the vortex. If αU^*_∞ is the order of magnitude of the velocity in the separation zone, the structure considered above, as applied to the flow model in [17], occurs in the case when $h^* L^{*-1} = O(\alpha \mathrm{Re}^{-1/2})$.

The streamline pattern for deck *3* when there are no velocity discontinuities is shown in Fig. 1(a) if the function (5.7) with $\delta_0 = 1$ is taken as the solution of the Benjamin–Ono equation (which describes the non-linear motion in this region). Solution (5.7) itself is shown in Fig. 1(b).

Figure 2(a) illustrates the configuration of the streamlines for the flow in deck *3* with a contact discontinuity (represented by the dashed curve) in a moving system of coordinates, where the corresponding solution of system of equations (4.3), (4.8), (4.9) and (7.3) is a steady-state solution. We have put $\rho = -2$ in (7.1), and the function $A = A - G$ (associated with the self-induced pressure in accordance with formula (3.10)) is shown in Fig. 2(b).

8. THE CHARACTERISTIC INSTABILITY OF THE DISCONTINUOUS SOLUTIONS OF THE ASYMPTOTIC EQUATIONS IN THE SHORT-WAVELENGTH LIMIT

The instability of a tangential discontinuity in a homogeneous medium with respect to infinitesimal perturbations [18] without allowing for the finite viscosity is such that the amplification factor of the perturbations increases without limit as the wave number increases.

We shall seek a solution of the system of equations

$$\begin{gathered}
\frac{\partial(A+G)}{\partial t} + (A+G)\frac{\partial(A+G)}{\partial x} = -\frac{\partial p}{\partial x} \\
\frac{\partial \Theta}{\partial t} + \Theta\frac{\partial \Theta}{\partial x} = -\frac{\partial p}{\partial x} \\
\frac{\partial G}{\partial t} + \frac{\partial}{\partial x}\left[G\Theta + \frac{\Omega_0}{2}G^2\right] = 0 \\
p = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial A(t,\xi)/\partial \xi}{\xi - x}\, d\xi
\end{gathered} \tag{8.1}$$

in the linear approximation. We put

$$(A, G, \Theta, p) = (A_0, G_0, \Theta_0, p_0) + (A', G', \Theta', p')ae^{ik(x-ct)} \tag{8.2}$$

where the amplitude factor a is assumed to be small. Considering short-wavelength perturbations, we assume that the quantitites A_0, G_0, Θ_0, p_0 are constant. Substituting expressions (8.2) into system (8.1) and neglecting terms of the order of a^2, we obtain a linear, homogeneous system of equations in A', G', Θ', p', and it follows from the fact that the determinant of this system is equal to zero that

$$\begin{aligned} k = {} & \frac{1}{2}\operatorname{sign}\Omega_0[(\Theta_0 - c)^2 + G_0(\tilde{A}_0 - c)]^{-1}(\Theta_0 - c)\times \\ & \times\{i\Omega_0 G_0 - (\Theta_0 - c)(\tilde{A}_0 - c) + [(i\Omega_0 G_0 + (\Theta_0 - c)(\tilde{A}_0 - c))^2 + \\ & + 4i\Omega_0 G_0(\tilde{A}_0 - c)^2]^{1/2}\}, \quad \tilde{A}_0 = A_0 + G_0 \end{aligned} \tag{8.3}$$

Expression (8.3) is obtained taking account of the equality

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ik\xi}}{\xi - x} d\xi = i \operatorname{sign} k e^{ikx} \tag{8.4}$$

which enables us to rewrite the last equation of system (8.1) in the form

$$p' = |k| A' \tag{8.5}$$

In (8.1)–(8.5), the wave number of the perturbations k is assumed to be real. As $|k| \to \infty$, we find from (8.3) that

$$c = k - i\Omega_0 G_0 / k \tag{8.6}$$

It is clear from (8.6) that inviscid instability (with respect to infinitesimal perturbations) of flows of the class under consideration with a line of tangential discontinuity is such that the imaginary part of the phase velocity tends to zero as k increases. In this sense, the behaviour of the perturbations being studied here is qualitatively different from the instability pattern of a contact discontinuity in an ideal fluid which is described by the complete Euler equations [18], where an increase in k is accompanied by an unlimited increase in the amplification factor of the perturbations. This substantial retardation in the growth of the perturbations is explained by the fact that, in system of equations (8.1), an interaction mechanism between the change in the shape of the line of discontinuity and the outer potential flow through the main deck of the boundary layer is established. In other words, even in the short-wavelength limit, model (8.1) remains non-local (in a direction along the normal to the surface around which the flow occurs).

Of course, a formal treatment (based on Euler's equations) of a small neighbourhood, containing a small part of the line of discontinuity and in which the flow may be considered as being homogeneous, leads to the identical conclusion [18] that a tangential discontinuity is always unstable. However, this treatment is meaningless as applied to the flow under investigation since, in the high wave number limit, the viscous structure of the contact discontinuity, which is, in fact, a mixing layer with a thickness of the order of $\Delta y = \mathrm{Re}^{-1/4}\alpha^{-2} \ll 1$, becomes important.

9. A QUADRUPLE-DECK ASYMPTOTIC SCHEME FOR FLOW AT TRANSONIC VELOCITIES OF THE OUTER STREAM

We will now extend the flow model, which has been introduced above, to the case of a transonic free stream from infinity $M_\infty^2 - 1 = O(\delta K_\infty)$, where $\delta \to 0$, $K_\infty = O(1)$. If, now, instead of (1.1) and (1.2), we consider perturbations in the main deck of the boundary layer of the form

$$\begin{aligned} &\frac{u^*}{U_\infty^*} = U_0 + \delta u_{1m} + \delta^2 u_{2m} + \ldots, \quad \frac{v^*}{U_\infty^*} = \delta^{5/4} v_{1m} + \delta^{7/4} v_{2m} + \ldots \\ &\frac{\rho^*}{\rho_\infty^*} = R_0 + \delta\rho_{1m} + \delta^2\rho_{2m} + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \delta^2 p_{1m} + \delta^3 p_{2m} + \ldots \end{aligned} \tag{9.1}$$

subject to the condition that the arguments T, X, Y_m of the required functions $u_{jm} = u_{jm}(T, X, Y_m)$, $\upsilon_{jm} = \upsilon_{jm}(T, X, Y_m)$, $\rho_{jm} = \rho_{jm}(T, X, Y_m)$, $p_{jm} = p_{jm}(T, X, Y_m)$ ($j = 1, 2, \ldots$) are defined by the expressions

$$t^* = \frac{L^*}{U_\infty^*}\left(1 + \frac{\varepsilon^4}{\delta^{5/2}} T\right), \quad x^* = L^*\left(1 + \frac{\varepsilon^4}{\delta^{3/2}} X\right), \quad y^* = L^* \varepsilon^4 Y_m \tag{9.2}$$

then the velocity field again acquires a multi-deck structure. The order relation $\delta = \varepsilon^{8/9}$ (where, as before, $\varepsilon = \mathrm{Re}^{-1/8}$) corresponds to the type of interaction with a transonic outer flow which was analysed for the first time in [19] and is characterized by a triple-deck pattern of the perturbed motion. Compared with the case when the Mach number differs from unity by a finite amount, the special feature of the theory in [19] is the fact that the interaction condition in it becomes dynamic (time dependent).

The increase in the amplitude of the pulsations, namely, the inequalities

$$\varepsilon^{8/9} \ll \delta \ll 1 \tag{9.3}$$

which are satisfied leads to the appearance of a quadruple-deck asymptotic structure, since the lower subdomain of the non-linear motion separates into an inviscid part and a viscous sublayer close to the wall, immediately adjacent to the surface. A boundary layer with a self-induced pressure in a transonic flow has been considered in [20] for the case when there are perturbations of comparatively large amplitude of the form of (9.1)–(9.3) in it. It was shown in [20] that the system of equations

$$\frac{\partial^2 \varphi}{\partial t \partial x} + K_\infty \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y_u^2} = 0 \tag{9.4}$$

$$\frac{\partial (A + G)}{\partial t} + (A + G)\frac{\partial (A + G)}{\partial x} = -\frac{\partial p}{\partial x} \tag{9.5}$$

$$\frac{\partial \varphi(t, x, 0)}{\partial x} = -p(t, x), \quad \frac{\partial \varphi(t, x, 0)}{\partial y_u} = -\frac{\partial A(t, x)}{\partial x} \tag{9.6}$$

holds in a special dimensionless system of units in the case of locally inviscid perturbations.

The potential $\varphi(t, x, y_u)$ of the outer velocity field obeys Eq. (9.4) in which the transonic parameter $K_\infty = O(1)$ characterizes the quantity $(M_\infty^2 - 1)\delta^{-1}$. The limit as $y_u \to 0$ corresponds to a transition from the potential part of the flow in the main deck of the boundary layer and relations (9.6) are the conditions for matching with (9.1) on its outer edge as $Y_m \to \infty$.

If the dependence of all the stream function on time and the spatial coordinate is defined by the self-similar variable $x - ct$, then, from (9.4), we have

$$(c - K_\infty)\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y_u^2} = 0 \tag{9.7}$$

Consequently, $\partial\varphi/\partial\eta_u + i\delta\varphi/\partial x$, where $\eta_u = (c - K_\infty)^{1/2} y_u$ is an analytic function of the complex variable $x + i\eta_u$. The Schwarz integral for the upper half plane, as it applies to this analytic function, can be written in the form

$$\frac{\partial \varphi(t, x, 0)}{\partial x} = -\frac{1}{\pi (c - K_\infty)^{1/2}} \int_{-\infty}^{\infty} \frac{\partial \varphi(t, x, 0)/\partial y_u}{\xi - x} d\xi \tag{9.8}$$

On substituting boundary conditions (9.6) into (9.8), we obtain an interaction condition which differs from (3.10) solely in the coefficient $(c - K_\infty)^{-1/2}$ on the right-hand side. In particular, the periodic solution of system of equations (9.4)–(9.6) is found by making the substitution

$$x \to x(c - K_\infty)^{1/2}, \quad t \to t(C - K_\infty)^{1/2}, \quad k \to k(c - K_\infty)^{-1/2}$$

in formula (7.10). Hence, all of the soliton-type solutions mentioned in Sections 4–7 (as well as those discussed in [21–23]) also hold in the case of the equations which describe the free interaction of a boundary layer with an outer transonic flow.

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